Nonequilibrium charge transport in quantum SINIS structures

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Charge transport in a high-transmission single-mode long SINIS junction (S stands for superconductor, I is an insulator, and N is a normal metal) is considered in the limit of low bias voltages and low temperatures. The kinetic equation for the quasiparticle distribution on the Andreev levels is derived taking into account both inelastic relaxation and voltage-driven Zener transitions between the levels. We show that, for a long junction when the number of levels is large, the Zener transitions enhance the action of each other and lead to a drastic increase of the dc current far above the critical Josephson current of the junction.

I. INTRODUCTION

Weak links consisting of superconducting (S) and normal metal (N) parts separated by insulators (I) are the subject of intensive experimental and theoretical studies. Transport through these systems is determined by several factors, such as transparency of the contacts, specifics of the weak link area and energy relaxation in the junction, leading to a series of nonlinear characteristics in the current-voltage relation. In stationary regime, numerous configurations have been analyzed (see Ref. [1] for review), revealing the importance of both Andreev[2] and normal reflection processes taking place in the contact.

When the injection rate of new particles into the junction is greater than the corresponding rate of inelastic relaxation the nonequilibrium effects are to be taken into account. In realistic junctions with high current density and especially at low temperatures the inelastic relaxation may become less effective, resulting in a strong nonequilibrium which crucially affects the current transported through the contact. For even a small bias voltage below the energy gap, the oscillating Josephson supercurrent may be accompanied with a nonzero dc component corresponding to the dissipative processes. Nonequilibrium situations in various SINIS-type junctions ranging from a point contact to a ballistic junction with finite length have been studied by many authors, the discussion to some extent has also concerned[3,4,5] the inelastic relaxation effects. As is well known, the dc component exhibits a subgap structure at bias voltages $eV = |\Delta|/n$ which is associated with multiple Andreev reflections (MAR)[3,6,7]. The quasiparticles trapped in the junction are accelerated by the applied voltage, while, for each cycle of repeated electron-hole reflections at the two NS interfaces, the energy of the particle increases by $2eV$ until the accumulated energy enables it to escape the pair-potential well. This works in a broad voltage range, but becomes more and more complicated when relaxation effects are included or the transparency at the interfaces differs from unity. The low-voltage MAR process for $eV \ll \Delta$ in a ballistic contact is equivalent to the spectral flow along the Andreev energy levels[8]

where the phase difference ϕ adiabatically depends on time $\phi = 2eV/\hbar + \phi_0$. However, for a non-ideal transparency, the energy levels are separated from each other by minigaps (see the next section) which suppress the transitions from one level to the next thus cutting the spectral flow off. As a result, for very low voltages, the dc current is small for contacts with any realistic transparency $\mathcal{T} \neq 1$.

The interlevel transitions can take place by means of Zener tunneling near the avoided crossings of the Andreev levels; they restore the spectral flow and give rise to a finite dissipative dc current. The Zener processes are more simple in short junctions (point contacts) where only two Andreev states corresponding to particles travelling in opposite directions exist; these levels have only one minigap at the phase difference $\phi = \pi$. Effects of Zener tunneling on the transport properties of quantum point contacts have been studied in Refs. [4,5]; the dc current was found to have an exponential dependence on voltage in the low-voltage limit.

For junctions where the center island has a length d longer than the superconducting coherence length ξ the number of levels is proportional to the ratio d/ξ and can, thus, become large. If the transparency is not exactly unity, these levels are separated by minigaps at $\phi = \pi k$, where k is an integer. In practice, such SINIS structures can be made of a carbon nanotube or semiconductor nanowire placed between two superconductors as in Refs. [9,10]. In Ref. [11] this type of junctions was suggested as a realization of a quantum charge pump where the minigaps were manipulated by the gate voltage being sequentially closed in resonance with the Josephson frequency. In the present paper we consider the low-temperature charge transport in these junctions for constant bias and gate voltages. We derive the effective kinetic equation for the quasiparticle distribution on the Andreev levels taking into account both the inelastic relaxation on each level and the Zener transitions between the neighboring levels and demonstrate that the voltage-driven Zener transitions from one level to the next enhance the action of each other and lead to a drastic increase of the dc current as the transition probability

grows with the applied voltage.

We begin with a brief description of the spectral properties of double-barrier SINIS structures in Section II. In Section III we derive the kinetic equation that determines the distribution function on the Andreev levels in the presence of Zener transitions and inelastic relaxation. In Sections IV and V we calculate the dc current and discuss the results.

II. MODEL

We consider a quantum SINIS contact consisting of two superconducting leads connected by a normal conductor that has a single conducting mode. The insulating barriers have a high transparency such that the contact is nearly ballistic. In this Section we briefly summarize the spectral properties of SINIS contacts which are important for the transport characteristics. It is well known that the supercurrent flowing through such contact is determined by the Andreev states formed in the normal conductor and extended into the superconducting leads. The states can be described by the Bogoliubov–deGennes equations

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_F + U(x) \right] \hat{\sigma}_z \hat{\psi} + \hat{H}\hat{\psi} = \epsilon\hat{\psi} , \quad (1)$$

where $\hat{\sigma}_z$ is the Pauli matrix in Nambu space, and

$$\hat{\psi} = \begin{pmatrix} u \\ v \end{pmatrix} , \ \hat{H} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} .$$

The superconducting gap is $\Delta = |\Delta|e^{\pm i\phi/2}$ for $x > d/2$ and $x < -d/2$, respectively, while $\Delta = 0$ for $-d/2 < x < d/2$. For simplicity we model the normal reflections at the interfaces as being produced by δ-function barriers $U(x) = I\delta(x - d/2) + I\delta(x + d/2)$ assuming that the quasiparticle velocity in the superconducting leads is the same as in the normal conductor.

In the normal region the particle, $e^{\pm iq_+ x}$, and hole, $e^{\mp iq_- x}$, waves have amplitudes $u^\pm$ and $v^\mp$, respectively. The upper or lower signs refer to the waves propagating to the right $\hat{\psi}^> = (u^+, v^-)$ or to the left $\hat{\psi}^< = (u^-, v^+)$. The particle (hole) momentum is $q_\pm = k_x \pm \epsilon/\hbar v_x$ where v_x is the quasiparticle velocity of the mode and $k_x = mv_x/\hbar$. Scattering at the right and left barriers couples the amplitudes of incident and reflected waves[12]:

$$\hat{\psi}_R^< = \hat{S}^R\hat{\psi}_R^>, \ \hat{\psi}_L^> = \hat{S}^L\hat{\psi}_L^<; \ \hat{S} = \begin{pmatrix} S_N e^{i\delta} & S_A e^{i\chi} \\ S_A e^{-i\chi} & S_N e^{-i\delta} \end{pmatrix}. \quad (2)$$

The scattering matrices for the right and left barriers are $\hat{S}^R = \hat{S}(\chi_R)$ and $\hat{S}^L = \hat{S}(\chi_L)$, respectively, where $\chi_L = -\phi/2$ while $\chi_R = \phi/2$. The scattering matrices are unitary $\hat{S}^\dagger\hat{S} = 1$ because of conservation of the quasiparticle flux. Components of the $\hat{S}$ matrix for δ-like barriers and energies $|\epsilon| < |\Delta|$ are[6]

$$S_N = -\frac{(U^2 - V^2)|Z|\sqrt{Z^2 + 1}}{U^2 + (U^2 - V^2)Z^2}, \ S_A = \frac{UV}{U^2 + (U^2 - V^2)Z^2}.$$

Here $Z = mI/\hbar^2 k_x$ is the barrier strength and $U = 2^{-1/2}[1 + i\sqrt{|\Delta|^2 - \epsilon^2}/\epsilon]^{1/2}$, $V = U^*$. The scattering phase δ is introduced through $\cot\delta = Z$. Applying the scattering conditions at both ends of the normal region one can derive a compact equation for the spectrum of a SINIS contact[11]

$$|S_N|^2 \sin^2\alpha' + |S_A|^2 \cos^2(\phi/2) = \sin^2(\beta + \gamma) . \quad (3)$$

Here $\alpha = k_x d$, $\alpha' = \alpha + \delta$, and $\beta = \epsilon d/\hbar v_x$. The phase γ is defined as $S_N = e^{i\gamma}|S_N|$. For short contacts $d \ll \hbar v_x/|\Delta|$, the spectrum has two branches varying from $\epsilon = \pm|\Delta|$ at $\phi = 0$ and separated from each other by minigaps at $\phi = \pi + 2\pi k$. The energy spectra of SINIS contacts in various limits have been extensively studied by many authors[13].
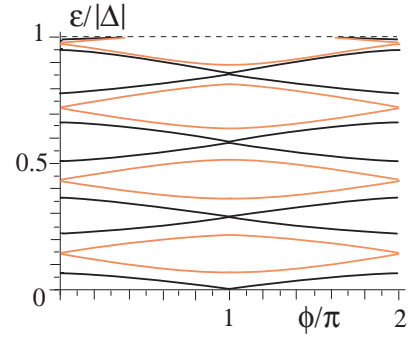

FIG. 1: (Color online) Examples of the spectra, Eq. (3), for a long SINIS contact with $Z = 0.5$ and $|\Delta|d/\hbar v = 10$. Dark (black online) lines: resonance $\sin\alpha' = 0$, the gaps disappear for $\phi = \pi + 2\pi k$; light (red online) lines: anti-resonance $\cos\alpha' = 0$, the gaps disappear for $\phi = 2\pi k$.

In what follows we focus on long contacts, $d \gg \hbar v_x/|\Delta|$ which have a large number of levels $N \sim d|\Delta|/\hbar v_x$. These levels split off from the states with $\epsilon = \pm|\Delta|$ and fill the energy interval $-|\Delta| < \epsilon < |\Delta|$ with spacings of the order of $\hbar v_x/d$. Examples of the spectra are shown in Fig. 1. Each level is a function of the phase difference ϕ; its range of variation is of the order of the interlevel spacing. The levels approach each other more closely at $\phi = \pi k$ where they are separated by minigaps. All minigaps at $\phi = \pi(1 + 2k)$ disappear for the resonance condition, $\sin\alpha' = 0$. Similarly, all minigaps at $\phi = 2\pi k$ disappear for anti-resonance, $|\sin\alpha'| = 1$. This follows from Eq. (3) due to unitarity $|S_N|^2 + |S_A|^2 = 1$. The low-energy levels, $\epsilon_l \ll |\Delta|$, $\gamma \ll 1$ have the form

$$\epsilon_l = \pm\epsilon_0 + \pi\hbar v_x l/d \quad (4)$$

where

$$\epsilon_0 = \frac{\hbar v_x}{d} \arcsin\sqrt{\mathcal{T}^2 \cos^2\frac{\phi}{2} + (1 - \mathcal{T}^2)\sin^2\alpha'} , \quad (5)$$

l is an integer, $\mathcal{T} = (1 + 2Z^2)^{-1}$ is the transmission coefficient of the contact. The energy gaps at $\phi = \pi(1 + 2k)$

are all equal,

$$\delta\epsilon_\pi = (2\hbar v_x/d)\arcsin\left[|\sin\alpha'|\sqrt{1-\mathcal{T}^2}\right] .$$

The gaps $\delta\epsilon_{2\pi}$ at $\phi = 2\pi k$ are given by the same expression where $|\sin\alpha'|$ is replaced with $|\cos\alpha'|$. For a transparent contact, $\mathcal{T} = 1$, all minigaps disappear.

III. KINETIC EQUATION

If a bias voltage V is applied across the superconducting leads, the current through the contact has both ac and dc components. For low voltages, eV much smaller than Δ, the dc current is small for contacts with any transparency $\mathcal{T} \neq 1$. This is due to the presence of minigaps discussed in the previous Section. In long contacts the minigaps exist at $\phi = \pi k$ and suppress the transitions between the levels thus preserving the equilibrium distribution of excitations. As a result, the current through the contact is simply the equilibrium supercurrent with the phase difference ϕ adiabatically depending on time, $\phi = \omega_J t + \phi_0$ where $\omega_J = 2eV/\hbar$ is the Josephson frequency. The dc component should thus vanish. However, the time dependence of the phase induces the Zener transitions between the levels near the avoided crossing points at $\phi = \pi k$ which produce deviation from equilibrium. Since the transparency of the contact is of the order of unity, $\mathcal{T} \sim 1$, particles with $|\epsilon| > |\Delta|$ have enough time to escape from the double-barrier region and to relax in the continuum. Particles at the continuum edges with energies $\epsilon = \pm|\Delta|$, which are in equilibrium with the heat bath, are captured on the outermost Andreev levels when the latter split off from the continuum as the phase varies in time. Next these particles are excited to the neighboring levels due to the interlevel transitions. Relaxation of thus created nonequilibrium distribution gives rise to a finite dc current.

In this Section we derive the effective kinetic equation that describes the distribution function on the Andreev levels taking into account both the inelastic relaxation on each level and the Zener transitions between the neighboring levels. We assume that the Zener transitions take place only near the avoided crossings at $\phi = \pi k$. This can be realized if two conditions are fulfilled. First, the minigaps $\delta\epsilon_\pi$ and $\delta\epsilon_{2\pi}$ should be much smaller than the average distance $\hbar v_x/d$ between the levels, implying that the contact is almost ballistic with a transparency close to unity, $1 - \mathcal{T} \ll 1$. Second, the applied voltage should be small $eV \ll \hbar v_x/d$ such that it cannot excite transitions between levels far from the avoided crossings. We also assume that, between the Zener tunneling events, the distribution function relaxes according to

$$\frac{\partial f_n}{\partial t} = -\frac{f_n - f_n^{(0)}}{\tau}$$

where τ is an inelastic relaxation time which we assume constant, and $f_n^{(0)} = 1 - 2n_{\epsilon_n}^{(0)} = \tanh(\epsilon_n/2T)$ is the

equilibrium distribution. Here n labels the levels consecutively from the lowermost level at $\epsilon = -|\Delta|$ up to the uppermost level at $\epsilon = +|\Delta|$. We will consider low temperatures $T \ll \hbar v_x/d$ such that

$$f_n^{(0)} = \text{sign}\,(\epsilon_n) . \qquad (6)$$

Since the superconducting phase difference depends linearly on time the distribution function can be written as $f_n(\phi) = f_n^{(0)} + \tilde{f}_n$ where

$$\tilde{f}_n(\phi) = f_n e^{-\hbar\phi/2eV\tau} . \qquad (7)$$

The amplitudes f_n of the decaying parts are to be found using the conditions at the transition points.
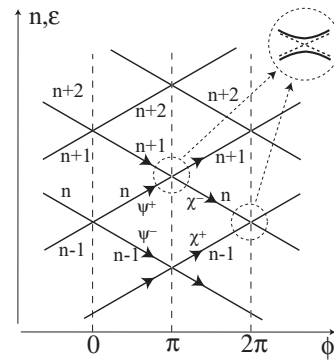

FIG. 2: Scheme of the energy levels as functions of ϕ. The arrows show the direction of the spectral flow. Enlarged are shown avoided crossings of levels at $\phi = 2\pi k$ and $\phi = \pi + 2\pi k$.

We assume that the interlevel transitions at $\phi = 2\pi k$ have a (constant) tunneling probability p_0 while the transitions at $\phi = \pi + 2\pi k$ have a probability p_π. We will see later that the assumption of constant probabilities is well justified. We denote the points $\phi = 2\pi k$ before and after the tunneling events as $0\mp$, respectively, while the points $\phi = \pi + 2\pi k$ before and after the tunneling events are denoted as $\pi\mp$. All the respective points $0\mp$ on a level n are equivalent due to the 2π periodicity, so are all the points $\pi\mp$. Using the evolution equation (7) we couple the distribution functions at the consecutive instants of the tunneling events (see Fig. 2)

$$f_n^\pm(\pi+) = f_n^{(0)} + \chi_n^\pm \quad , \quad f_n^\pm(0-) = f_n^{(0)} + \chi_n^\pm e^{-\nu} ,$$
$$f_n^\pm(0+) = f_n^{(0)} + \psi_n^\pm \quad , \quad f_n^\pm(\pi-) = f_n^{(0)} + \psi_n^\pm e^{-\nu} ,$$

where $\nu = \pi\hbar/2eV\tau$. Here we introduce the upper $+$ (or $-$) indices to indicate explicitly the distributions at the spectrum branches increasing (or decreasing) as functions of ϕ. The coefficients $\phi_n^\pm$ and $\chi_n^\pm$ are the amplitudes f_n in Eq. (7) defined for the intervals $0 < \phi < \pi$ and $\pi < \phi < 2\pi$, respectively.

The tunneling events impose the relations

$$f_n^+(0+) = p_0 f_{n-1}^+(0-) + (1-p_0)f_n^-(0-) , \quad (8)$$
$$f_n^-(\pi+) = p_\pi f_{n+1}^-(\pi-) + (1-p_\pi)f_n^+(\pi-) , \quad (9)$$
$$f_{n+1}^-(0+) = p_0 f_{n+2}^-(0-) + (1-p_0)f_{n+1}^-(0-) , \quad (10)$$
$$f_{n+1}^+(\pi+) = p_\pi f_n^+(\pi-) + (1-p_\pi)f_{n+1}^-(\pi-) , \quad (11)$$

illustrated in Fig. 2. For the coefficients ψ and χ they become

$$\psi_n^+ = p_0[f_{n-1}^{(0)} - f_n^{(0)}]$$
$$+[p_0\chi_{n-1}^+ + (1-p_0)\chi_n^-]e^{-\nu}, \quad (12)$$
$$\psi_{n+1}^- = p_0[f_{n+2}^{(0)} - f_{n+1}^{(0)}]$$
$$+[p_0\chi_{n+2}^- + (1-p_0)\chi_{n+1}^+]e^{-\nu}, \quad (13)$$

and

$$\chi_n^- = p_\pi[f_{n+1}^{(0)} - f_n^{(0)}] + [p_\pi\psi_{n+1}^- + (1-p_\pi)\psi_n^+]e^{-\nu}, (14)$$
$$\chi_{n+1}^+ = p_\pi[f_n^{(0)} - f_{n+1}^{(0)}] + [p_\pi\psi_n^+ + (1-p_\pi)\psi_{n+1}^-]e^{-\nu}. (15)$$

According to our picture of the spectral flow through the Andreev levels, the boundary conditions are imposed at the continuum edges in such a way that, for the levels increasing as functions of ϕ, the distribution f^+ coincides with the equilibrium at $\epsilon = -|\Delta|$, while, for decreasing levels, the distribution f^- coincides with the equilibrium at $\epsilon = +|\Delta|$. Since the bias voltage is low, $eV \ll \hbar v_x/d$, the equilibrium function in both superconducting electrodes can be taken as $f^{(0)} = \tanh(\epsilon_n/2T)$. Let us assume that trapping of particles from the continuum occurs at $\phi = 0$; the boundary conditions are then formulated for the function ψ_n^+ at $\epsilon = -|\Delta|$ and for ψ_n^- at $\epsilon = +|\Delta|$:

$$\psi_{\epsilon=-|\Delta|}^+ = 0 , \quad \psi_{\epsilon=+|\Delta|}^- = 0 . \quad (16)$$

In this case it is convenient to exclude the functions χ using Eqs. (14) and (15) and solve Eqs. (12), (13) for the functions ψ.

We choose the level index n in such a way that $\epsilon_n > 0$ for $n \geq 1$ and $\epsilon_n < 0$ for $n \leq 0$. Equations (12) and (13) then couple the levels n and $n \pm 2$. Since the temperature is low and the distribution is given by Eq. (6), the r.h.s. of these equations vanish for all $n \neq 0, 2$. Therefore, the coefficients for $n \geq 2$ and $n \leq 0$ satisfy the homogeneous equations. We assume that there are $N+1$ levels with positive energies and $N+1$ levels with negative energies such that the outermost levels touch the continuum. Therefore, for the uppermost level $n = N + 1$, the solution of Eqs. (12), (13) satisfies the condition $\psi_{N+1}^- = 0$. Similarly, for the lowermost level $n = -N$ the solution satisfies $\psi_{-N}^+ = 0$. With these conditions, the solutions for $n \geq 2$ are

$$\psi_n^+ = c^> \left[e^{-r(N-n+1)}w_- - e^{r(N-n+1)}w_+ \right] ,$$
$$\psi_n^- = c^> \left[e^{-r(N-n+1)} - e^{r(N-n+1)} \right] . \quad (17)$$

where

$$w_\pm = \frac{p_0 e^{\pm r} + p_\pi e^{\mp r} - 2p_0 p_\pi \cosh(r)}{\zeta + p_0 p_\pi(1 - e^{\pm 2r}) + p_0 + p_\pi - 2p_0 p_\pi} .$$

The solutions for $n \leq 0$ have the form

$$\psi_n^+ = c^< \left[e^{r(N+n)} - e^{-r(N+n)} \right] ,$$
$$\psi_n^- = c^< \left[e^{r(N+n)}w_+ - e^{-r(N+n)}w_- \right] . \quad (18)$$

The effective relaxation rate $r > 0$ is found from the determinant condition $w_+ w_- = 1$ which gives

$$4p_0 p_\pi(\zeta + 1)\sinh^2(r) = (\zeta + p_0 + p_\pi - 2p_0 p_\pi)^2$$
$$- (p_0 + p_\pi - 2p_0 p_\pi)^2 . \quad (19)$$

where $\zeta = e^{2\nu} - 1$.

For an ideally transparent contact $p_0 = p_\pi = 1$ we find $r = \nu$. For strong relaxation, $\nu \gg 1$, we also have $r \approx \nu$, and the distribution relaxes quickly. The most interesting limit for a general case $p_0, p_\pi \neq 1$ is when inelastic relaxation is weak, $\nu \ll 1$. We find in this limit

$$\sinh^2(r) = \nu(\nu + p_0 + p_\pi - 2p_0 p_\pi)/p_0 p_\pi . \quad (20)$$

The relaxation rate r can be either large or small depending on the probabilities. The inverse rate r^{-1} describes the broadening of distribution over the energy states and plays the role of an effective temperature $T_{\text{eff}} = r^{-1}(d\epsilon/dn) = \pi\hbar v_x/2rd$. The effective temperature can be much higher than the interlevel spacing if $r \ll 1$.

The coefficients $c^>$ and $c^<$ are coupled through the solutions of four non-homogeneous equations resulting from two equations (12) and (13) taken for two values $n = 2$ and $n = 0$. Inspecting equations for other n we see that only the two coefficients ψ_0^+ and ψ_1^- cannot be described by the solutions Eqs. (17) and (18) of the homogeneous equations. We write

$$\psi_1^- = \psi_1^{-> } + \delta_1^> , \quad \psi_1^- = \psi_1^{-<} + \delta_1^< , \quad (21)$$
$$\psi_0^+ = \psi_0^{+>} + \delta_0^> , \quad \psi_0^+ = \psi_0^{+<} + \delta_0^< . \quad (22)$$

Here $\delta_{0,1}^>$ and $\delta_{0,1}^<$ are four new unknown coefficients. The coefficients $\psi_1^{->}$ and $\psi_0^{+>}$ are defined to satisfy the homogeneous equations for $\epsilon_n > 0$ and are given by Eq. (17); the coefficients $\psi_1^{-<}$ and $\psi_0^{+<}$ satisfy the homogeneous equations for $\epsilon_n < 0$ and are given by Eq. (18). Inserting Eqs. (21), (22) into the four equations obtained for $n = 2$ and $n = 0$ from Eqs. (12) and (13) we find all the four coefficients δ. The result is $\psi_1^- = \psi_1^{->}$, $\psi_0^+ = \psi_0^{+<}$ while

$$\psi_1^{-<} - \psi_1^{->} = e^\nu(f_1^{(0)} - f_0^{(0)}) , \quad (23)$$
$$\psi_0^{+<} - \psi_0^{+>} = e^\nu(f_1^{(0)} - f_0^{(0)}) . \quad (24)$$

These two equations yield $c^> = c^< = C$ where

$$C \left[e^{rN}(1 + e^r w_+) - e^{-rN}(1 + e^{-r}w_-) \right] = 2e^\nu . \quad (25)$$

We put $f_1^{(0)} - f_0^{(0)} = 2$ for low temperatures. Therefore, the distribution possesses the symmetry $\psi_{-n}^+ = -\psi_{n+1}^-$.

IV. CURRENT

The contribution to the current due to the deviation from equilibrium is

$$I_{\text{neq}} = -\frac{2e}{\hbar} \sum_{\epsilon_n > 0} \frac{\partial \epsilon_n}{\partial \phi} \tilde{f}_n \qquad (26)$$

where $\tilde{f}_n = f_n - f_n^{(0)}$. The sum runs only over the localized Andreev states because the continuum states relax quickly so that their distribution is almost in equilibrium. The equilibrium supercurrent has been calculated in Ref. [14] (see also Ref. [1] for a review); it is an oscillating function of the phase difference and thus has no contribution to the dc current.

Denote $\partial \epsilon_n^\pm / \partial \phi$ increasing (decreasing) parts of the spectrum $\epsilon_n(\phi)$ as a function of ϕ. We have for the current averaged in time

$$\overline{I} = -\frac{e}{\pi\hbar} \sum_{l=0}^{N/2} \left[\int_0^\pi \left(\frac{\partial \epsilon_{n+1}^+}{\partial \phi} \tilde{f}_{n+1}^+ + \frac{\partial \epsilon_n^-}{\partial \phi} \tilde{f}_n^- \right) d\phi \right.$$
$$\left. + \int_\pi^{2\pi} \left(\frac{\partial \epsilon_{n+1}^-}{\partial \phi} \tilde{f}_{n+1}^- + \frac{\partial \epsilon_n^+}{\partial \phi} \tilde{f}_n^+ \right) d\phi \right]_{n=2l+1} . (27)$$

The sum over l runs from 0 to $L = N/2$ where

$$N = 2d\Delta / \pi\hbar v_x$$

is the total number of levels with $\epsilon_n > 0$, i.e., for both signs in Eq. (4) .

In Eq. (27) we can use the ballistic spectrum with $\mathcal{T} \to 1$. In this limit $|S_N| = 0$, $|S_A| = 1$, thus the spectrum in Eq. (3) takes the form

$$\cos(\phi/2) = \pm \sin(\beta + \gamma) .$$

Calculating the energy derivative of this equation for long junctions $d|\Delta| \gg \hbar v_x$, we find

$$\frac{\partial \epsilon_n^\pm}{\partial \phi} = \pm \frac{\hbar v_x}{2d} .$$

We neglected $\partial \gamma / \partial \epsilon$ compared to $\partial \beta / \partial \epsilon$ which holds for all energy levels excluding those in a narrow region near the gap edge, $1 - |\epsilon|/|\Delta| \ll (\hbar v_x/d|\Delta|)^2$. This means in fact that, neglecting this narrow region, we can use Eqs. (4) and (5) with $\mathcal{T} = 1$ for all n.

We have for $0 < \phi < \pi$

$$\tilde{f}_n^\pm = \psi_n^\pm e^{-\hbar\phi/2eV\tau} ,$$

while for $\pi < \phi < 2\pi$

$$\tilde{f}_n^\pm = \chi_n^\pm e^{-\hbar(\phi-\pi)/2eV\tau} .$$

We obtain for $\nu \ll 1$

$$\overline{I} = \frac{ev_x}{2d} \Phi(p_0, p_\pi) = \frac{\pi\hbar v_x}{2eR_0 d} \Phi(p_0, p_\pi) \qquad (28)$$

where $R_0^{-1} = e^2/\pi\hbar$ is the quantum of conductance, and

$$\Phi(p_0, p_\pi) = -\sum_{l=0}^{N/2} \left[\left(\psi_{n+1}^+ - \psi_n^- \right) + \left(\chi_n^+ - \chi_{n+1}^- \right) \right]_{n=2l+1} .$$

The combination of coefficients ψ and χ that enters the expression for the supercurrent for $l \geq 1$ can be written through the solutions Eq. (17) of the homogeneous equations (12)–(15). The term $l = 0$ contains χ_1^+ which is expressed through ψ_0^+. However, one can check that the jump in ψ_n^+ from $n = 0$ to $n = 2$ is compensated by the jump $f_1^{(0)} - f_0^{(0)}$ in Eq. (15).

Consider the limit of low relaxation $r \ll 1$ provided $Nr \gg 1$. The limit $r \ll 1$ is realized when $\nu \ll p_0, p_\pi$. In this case Eqs. (15), (17), and (25) give

$$(\psi_{n+1}^+ - \psi_n^-) + (\chi_n^+ - \chi_{n+1}^-) = -\frac{4\nu}{r} e^{-rn} . \qquad (29)$$

We have from Eq. (29)

$$\Phi(p_0, p_\pi) = \frac{4\nu}{r} \sum_{k=0}^{N/2} e^{-r(2k+1)} = \frac{2p_0 p_\pi}{p_0 + p_\pi - 2p_0 p_\pi + \nu} . \qquad (30)$$

We keep ν in the denominator since the combination $p_0 + p_\pi - 2p_0 p_\pi$ vanishes when $p_0, p_\pi \to 1$.

When the inelastic relaxation rate is so small that $N\nu \ll 1$ the effective relaxation r can decrease such that $Nr \ll 1$. Since the product $N(p_0 + p_\pi - 2p_0 p_\pi)$ is generally not small we find from Eqs. (14), (15), (17), and (25)

$$(\psi_{n+1}^+ - \psi_n^-) + (\chi_n^+ - \chi_{n+1}^-) = -\frac{4p_0 p_\pi}{N(p_0 + p_\pi - 2p_0 p_\pi) + 1},$$

and

$$\Phi(p_0, p_\pi) = \frac{2p_0 p_\pi}{(p_0 + p_\pi - 2p_0 p_\pi) + 1/N} . \qquad (31)$$

Eqs. (30) and (31) go one into another for $N \sim 1/\nu$. The exact expression for Φ is found from Eqs. (14), (15), (17), and (25). One can approximate the function $\Phi(p_0, p_\pi)$ by an interpolation between Eqs. (30) and (31) in the form

$$\Phi(p_0, p_\pi) = \frac{2p_0 p_\pi}{(p_0 + p_\pi - 2p_0 p_\pi) + 2\beta} \qquad (32)$$

where $2\beta \approx \max(\nu, N^{-1}) \approx \nu + N^{-1}$.

The probability of Zener tunneling can be easily calculated for the spectrum in the form of Eqs. (4), (5) if $(1 - \mathcal{T}^2) \ll 1$. The phase difference is $\phi = \omega_J t + \phi_0$. As a function of time, the distance between two neighboring levels for ϕ close to π is

$$\delta\epsilon = \frac{\hbar v_x \mathcal{T} \omega_J}{d} \sqrt{t^2 + \tau_0^2}$$

where t is small and

$$\tau_0^2 = 4\sin^2 \alpha' (1 - \mathcal{T}^2)/\mathcal{T}^2 \omega_J^2 .$$

Probability of Zener tunneling is

$$p_\pi = \exp\left[-\frac{2}{\hbar}\,\mathrm{Im}\left(\int_0^{i\tau_0}\delta\epsilon\,dt\right)\right] = \exp\left[-\frac{\omega_0}{\omega_J}\sin^2\alpha'\right]$$

where

$$\omega_0 = \pi v_x(1-\mathcal{T}^2)/\mathcal{T}d\ .$$

The distance between two levels for ϕ close to $\phi = 0$ and the corresponding probability of Zener tunneling p_0 are given by the same expressions with $\sin\alpha'$ replaced with $\cos\alpha'$.

In Eq. (32) the term with β is only important when p_0 and p_π are close to unity. Therefore, one can write

$$\Phi(p_0,p_\pi) = \left(\exp\left[\frac{\omega_0}{\omega_J}\right]\cosh\left[\frac{\omega_0}{\omega_J}\cos(2\alpha')\right] - 1 + \beta\right)^{-1}.$$
(33)

When the bias voltage is low $\omega_J/\omega_0 \lesssim 1$, such that $\nu \ll p_0, p_\pi \ll 1$, we have

$$\Phi(p_0,p_\pi) = 2\exp\left(-\frac{\omega_0}{\omega_J}[1 + |\cos(2\alpha')|]\right)\ .$$
(34)

The low-voltage part is exponential due to small Zener probabilities. The exponent exhibits strong oscillations as a function of α' which can be manipulated by varying the gate voltage.

For higher bias voltages, $\omega_J/\omega_0 \gg 1$, we have $p_0, p_\pi \to 1$ and

$$\Phi(p_0,p_\pi) = \frac{\omega_J}{\omega_0 + \omega_J\beta} \approx \frac{\omega_J}{\omega_0 + \pi/2\tau + \omega_J/2N}\ .$$

For a fully ballistic contact with $p_0 = p_\pi = 1$, i.e., $\omega_0 = 0$ our result agrees qualitatively with Ref. [3]. For these voltages we have two regimes. The I-V curve is linear $I = V/R$ as long as $eV \ll N(\hbar\omega_0 + \pi\hbar/\tau)$. The effective conductance is

$$\frac{1}{R} = \frac{1}{R_0}\frac{\pi v_x/d}{\omega_0 + \pi/2\tau}\ .$$

Inelastic relaxation can be neglected if $\omega_0 \gg \pi/2\tau$. In this case the conductance is much larger than the conductance quantum $R_0/R = (1-\mathcal{T}^2)^{-1}$. It is interesting to note that the effective conductance is independent of the gate voltage: it contains the sum of two functions in the exponents for p_0 and p_π, i.e., $(\omega_0/\omega_J)\cos^2\alpha'$ and $(\omega_0/\omega_J)\sin^2\alpha'$, which obviously is independent of α'.

With increasing voltage up to $eV \gtrsim N(\hbar\omega_0 + \pi\hbar/\tau)$ the I-V curve saturates at the value

$$I = Nev_x/d = 2e|\Delta|/\pi\hbar$$

which is by a factor $N \gg 1$ larger than the critical Josephson current of the junction[14], $I_c \sim ev_x/d$.

V. CONCLUSIONS

To summarize, we have considered the low temperature charge transport in a nearly ballistic single-mode SINIS junction having a length d longer than the superconducting coherence length ξ. In this junction the energy spectrum of Andreev states has a large number of levels separated from each other by minigaps which do not vanish in a realistic case when the transmission is not exactly unity. In the limit of low bias voltages, we have derived and solved the kinetic equation for the quasiparticle distribution on the Andreev levels that takes into account both inelastic relaxation and voltage-driven Zener transitions between the levels. We have shown that the Zener transitions enhance the action of each other and lead to a drastic increase of the dc current. The voltage dependence of the dc current is first exponential due to small probabilities of Zener tunneling. Next it goes over into a linear relation such that, at low temperatures when the inelastic relaxation rate is slow, its slope is determined by the average minigap in the spectrum. At higher voltages when the Zener probabilities approach unity, the dc current saturates at a value far exceeding the critical Josephson current of the junction.

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